Differentiation

Techniques

- rules: sum, product, quotient, chain, constant, coefficient
- implicit differentiation: differentiate $y$ with the same rules as if it were an $x$ but remember to multiply by the “derivative of the inside” which is $\frac{dy}{dx}$
- logarithmic differentiation: take ln of both sides use rules of logs to rewrite the right side differentiate recalling that the derivative of $\ln(f(x))$ is $\frac{f'(x)}{f(x)}$ solve for $f'(x)$ and replace $f(x)$ with its proper expression
- partial derivatives: derivatives for two variable functions use the same rules as single-variable functions because we always differentiate with respect to one variable at a time and treat the other as a constant

Applications

- slope of tangent line to a curve (historically this is how derivatives were discovered/invented)
- approximation by increments: $f(x) - f(x_0) \approx f'(x_0)(x - x_0)$
- marginal cost (or marginal profit, etc.)
- related rates (uses implicit diff)
- applied optimization (see optimization)
- optimization and graphing: critical points ($f'(cp) = 0$ or undefined) intervals of incr/decr ($f'$ is $+/-$, resp.) concavity ($f''$ is $+/-$ $\Rightarrow f$ is CU/CD) inflection points ($f''$ changes sign and $f$ must be defined at inflection points) relative max/min (see derivative tests)
- 1st derivative test: $cp$ where $f'$ changes from $+$ to $-$ (resp. - to $+$) is max (resp. min) as long as $f$ is defined at said $cp$
- 2nd derivative test: $cp$ where $f''$ is negative (resp. positive) is a max (resp. min) as long as $f$ is defined at said $cp$
- global (absolute) max/min: on an open interval a local max/min is a global max/min if it is the only $cp$; on a closed interval look at values of $f$ at critical points and endpoints to determine the global max (largest) and min (smallest)
- vert./horizontal asymptote (see limits) in higher dimensions, critical points make both $f_x$ and $f_y$ zero; classify $cp$ by looking at $D(cp) = f_{xx}(cp)f_{yy}(cp) - f_{xy}(cp)^2$; if $D(cp) < 0$, $cp$ is a saddle point; if $D(cp) > 0$ then $cp$ is a max (resp. min) if $f_{xx}(cp) < 0$ (resp. $> 0$)
Integration

Techniques

• similar sum and coefficient rules as derivative but different product and chain rules

• indefinite integrals must have a “+C”

• FTOC: If \( F' = f \), then \( \int_a^b f(x) \, dx = F(b) - F(a) \)

• substitution
  choose \( u \) and find \( du \)
  substitute the \( du \) and write everything else in terms of \( u \)
  if definite integral, change the limits to be with respect to \( u \)
  if indefinite integral, be sure to put your answer back in terms of the original variable

• integration by parts
  choose \( u \) and let everything else be \( dv \) (\( u \) will be \( \ln x \) or \( x^n \) usually)
  differentiate \( u \) to get \( du \)
  integrate \( dv \) to get \( v \)
  plug into the formula \( uv - \int v \, du \) and finish the computation

• double integrals
  make sure inside limits of integration are with respect to inside variable
  integrate the inside integral using one variable rules by treating the other variable as a constant
  evaluate and simplify the integrand
  you should be left with an integral in one variable
  solve normally
  double integrals may require techniques like substitution

Applications

• find the area under the curve (historically this is how integrals were discovered/invented)

• initial value problem (given a rate, integrate it; then use initial value to determine what “+C” is)

• net change (definite integral of a rate)

• area between curves (integrate the quantity (largest function minus smallest function); set functions equal to find limits of integration, if necessary)

• average value (area under curve times \( \frac{1}{b-a} \))

• volume under the surface (double integral)
Limits

Techniques

- at a point:
  plug in the point
  \( L \Rightarrow L \)
  \( \frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty \Rightarrow \text{factor or rationalize} \)
  \( \frac{L}{0} \Rightarrow \text{type of infinity, plug in nearby point to determine sign} \)

- at \( \pm \infty \):
  locate largest exponent
  in the denominator \( \Rightarrow 0 \)
  in the numerator \( \Rightarrow \text{type of } \infty \), determine sign by looking at largest terms in numerator and denominator
  same power in both \( \Rightarrow \text{ratio of leading coefficients} \)

- recall the behavior of \( e^x \) and \( \ln x \):
  \( \lim_{x \to -\infty} e^x = 0, \lim_{x \to \infty} e^x = +\infty \)
  \( \lim_{x \to 0^+} \ln x = -\infty, \lim_{x \to \infty} \ln x = +\infty \)

Applications

- continuity: \( \lim_{x \to a} f(x) \) exists and equals \( f(a) \)
- definition of the derivative: \( \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \)
- vertical asymptotes: determine point where the limit has the form \( \frac{L}{0} \)
- horizontal asymptotes: occur if \( \lim_{x \to -\infty} f(x) \) or \( \lim_{x \to \infty} f(x) \) is finite

Miscellaneous

Techniques

- profit, revenue, cost, average cost
- \( Pe^{rt} \)